

3.1. LINEAR MODELS

GROW AND DECAY

$$\frac{dx}{dt} = Kx, \quad x(t_0) = x_0 \quad (1)$$

Example 1.

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacteria is measured to be $\frac{3}{2} P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t , determine the time necessary for the number of bacteria to triple.

Solution:

$$\frac{dP}{dt} - KP = 0$$

Integrating factor $\mu(t) = e^{\int -K dt} = e^{-Kt}$.

$$d[\mu P] = 0$$

$$\Leftrightarrow e^{-Kt} P = c \Leftrightarrow P = c \cdot e^{Kt}.$$

We need to find c and K .

$$\boxed{t=0:}$$

$$P_0 = C$$

$$\boxed{t=1}$$

$$\frac{3}{2} P_0 = P_0 \cdot e^k$$

$$\Rightarrow e^k = \frac{3}{2} \Rightarrow k = \ln \frac{3}{2} (\approx 0.4055)$$

We need to find t such that

$$P(t) = P_0 e^{(\ln \frac{3}{2})t} = 3P_0$$

$$\Rightarrow e^{(\ln \frac{3}{2})t} = 3$$

$$\Rightarrow \ln \frac{3}{2} t = \ln 3$$

$$\Rightarrow t = \frac{\ln 3}{\ln \frac{3}{2}} \approx 2.71 \text{ (h).}$$

□

HALF LIFE. In physics the **half-life** is a measure of the stability of a radioactive substance. This is simply the time it takes for $\frac{1}{2}$ of the atoms in the initial amount A_0 to disintegrate, or transmute, into the atoms of another element.

Example. A breeder reactor converts relatively stable uranium-238 into isotope plutonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate

of disintegration is proportional to the amount remaining.

Solution:

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

$$A(t) = A_0 e^{kt}$$

If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. We can find k :

$$\Rightarrow 0.99957 A_0 = A_0 e^{15k}$$

$$\Rightarrow k = \frac{1}{15} \ln 0.99957 \approx -0.00002867$$

Find the half-life t :

$$\frac{1}{2} A_0 = A_0 e^{-0.00002867 t}$$

$$\Rightarrow \frac{1}{2} = e^{-0.00002867 t}$$

$$\Rightarrow t = \frac{\ln \frac{1}{2}}{-0.00002867} = \frac{\ln 2}{0.00002867} \approx 24,180 \text{ yr.}$$

□

Reading exercise: (Example about Age of Fossil)

NEWTON'S LAW OF COOLING/WARMING

$$\frac{dT}{dt} = K(T - T_m)$$

where T_m is the temperature of the medium around the object.

Example 4. When a cake removed from an oven, its temperature is measured at 300°F . Three minutes later the temperature is 200°F . How long will it take for the cake to cool off to 100°F ? The room temper. is 70°F .

Solution:

$$\frac{dT}{dt} = K(T - T_m), \quad T(0) = 300$$

$$\Rightarrow \frac{dT}{(T-70)} = K dt \quad (\text{separable eq.})$$

$$\Rightarrow \ln|T-70| = Kt + C_1$$

$$\Rightarrow T - 70 = C e^{Kt}$$

$$\Rightarrow T = C \cdot e^{Kt} + 70$$

$$\boxed{t=0} \quad 300 = C + 70 \Rightarrow C = 230$$

$$\boxed{t=3} \quad 200 = 230 \cdot e^{3K} + 70$$

$$\Rightarrow e^{3K} = \frac{130}{230} \Rightarrow K = \frac{1}{3} \ln \frac{130}{230} \approx -0.19018$$

Find t

$$100 = 70 + 230 e^{-0.19018t}$$

$$\Rightarrow e^{-0.19018t} = \frac{30}{230}$$

$$\Rightarrow -0.19018t = \ln \frac{3}{23}$$

$$\Rightarrow t = \frac{\ln \frac{23}{3}}{0.19018} \approx 10.7 \text{ (mins)}$$

□

MIXTURE:

Example: A large tank holds 200 gals of brine solution. Brine is being pumped into the tank at the rate 4 gal/min. The concentration of the salt in the inflow is 3 lb/gal. The mixture is also pumped out at the rate 4 gal/min.

$$R_{in} = (4 \text{ gal/min}) (3 \text{ lb/gal}) = 12 \text{ lb/min}$$

$$\begin{aligned} R_{out} &= \left(\frac{A}{200} \text{ lb/gal} \right) (4 \text{ gal/min}) \\ &= \frac{A}{50} \text{ lb/min} \end{aligned}$$

Assume that 50 pounds of salt were dissolved initially in the 200 gallons, how much salt in the tank after a long time?

Solution:

Recall: $\frac{dA}{dt} = R_{in} - R_{out}$

$$\frac{dA}{dt} = 12 - \frac{A}{50}$$

$$\Leftrightarrow \frac{dA}{dt} + \frac{A}{50} = 12, \quad A(0) = 50.$$

Integrating factor is $e^{t/50}$

$$\Rightarrow \frac{d}{dt} [A e^{t/50}] = 12 e^{t/50}$$

$$\Rightarrow A \cdot e^{t/50} = 600 e^{t/50} + C.$$

$$\Leftrightarrow A = 600 + C e^{-t/50}.$$

$$\boxed{t=0} \quad 50 = 600 + C.$$

$$\Leftrightarrow C = -550$$

$$\text{So } A(t) = 600 - 550 e^{-t/50}.$$

$$\text{Let } t \rightarrow \infty, \quad A(t) \rightarrow 600. \quad \square$$

It is as we expect, as after a long time the concentration of solution in the tank is equal to the concentration of inflow.

$$\Rightarrow \text{We have total } (200 \text{ gal}) (3 \text{ lb/gal}) = 600 \text{ lb.}$$

Example 6 We now assume that the mixture was pumped out at a slower rate of 3 gal/min. Then the liquid will accumulate in the tank at rate

$$(4-3) \text{ gal/min} = 1 \text{ gal/min}$$

$$\text{new } R_{\text{out}} = \left(\frac{A}{200+t} \right) \cdot 3 = \frac{3A}{200+t} \text{ (lb/min)}$$

→ our eq becomes

$$\frac{dA}{dt} = 12 - \frac{3A}{200+t}$$

$$\Rightarrow \frac{dA}{dt} + \frac{3}{200+t} A = 12$$

$$\text{integrating factor is } e^{\int (3/200+t) dt} = (200+t)^3$$

$$\Rightarrow \frac{d}{dt} \left((200+t)^3 A \right) = 12 (200+t)^3$$

$$\Rightarrow (200+t)^3 A = 3 (200+t)^4 + C$$

$$\Rightarrow A = 3(200+t) + C / (200+t)^3$$

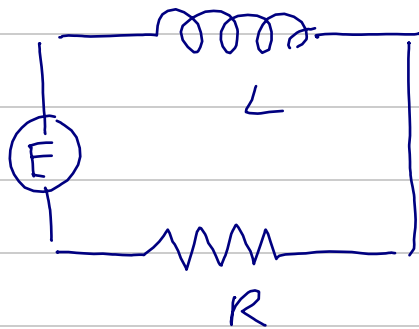
$$\boxed{t=0} : 50 = 600 + \frac{C}{200^3}$$

$$C = - \frac{8 \cdot 10^6}{550}$$

$$\Rightarrow A(t) = 600 + 3t - \frac{\frac{800 \cdot 10^6}{550}}{(200+t)^3} \rightarrow \infty \text{ as } t \rightarrow \infty.$$

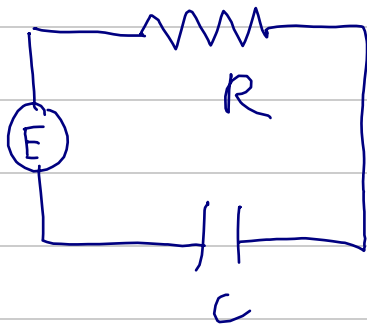
SERIES CIRCUITS.

Consider an LR-series circuit:



By Kirchhoff's second law we have

$$L \frac{di}{dt} + Ri = E(t) \quad (7)$$



Similar an RC-series circuit give

$$R \frac{dq}{dt} + \frac{1}{C} q = E(t) \quad (8)$$

Example 7. A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is 0.

Solution. $\frac{1}{2} \frac{di}{dt} + 10i = 12, \quad i(0) = 0$

Integrating factor $\mu(t) = e^{20t}$

$$\frac{d}{dt} [e^{20t} i] = 24 e^{20t}$$

$$\Rightarrow i(t) = \frac{6}{5} + C \cdot e^{-20t}$$

$$\boxed{i=0} \quad C = -\frac{6}{5}.$$

$$\Rightarrow i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t} \quad \square$$